The Definitions of Number in Boethius's *Introduction to Arithmetic*

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Abstract: The paper enquires into the reasons why Boethius altered the passage addressing the definition of number in his loose translation of Introduction to Arithmetic by the Neopythagorean philosopher and mathematician Nicomachus of Gerasa. While Nicomachus's text contains three definitions of number, Boethius lists only two. However, he also pays attention to the definition he omits, even though he does not regard it as a proper definition. In his view it fails to embody the essence of number, and is to be understood as a description of the components constitutive of the substance of number. Although this is a possible explanation of Boethius's dismissal of the definition provided by Nicomachus, the description also occupies an important place in relation to the general characteristic of number, because Nicomachus's definitions fully correspond to the three basic topics which were central to contemporary arithmetic, viz. the science of number: number as discrete quantity, referring to the properties of numbers and their classifications; number as collection of units, leading to the topic of figural numbers; and number as quantity emanating from unit and subsequently returning to it, corresponding with numerical ratios, sequences and their transfers.

Keywords: Boethius; Nicomachus of Gerasa; number; arithmetic; quadrivium.

1. Introduction

Medieval arithmetic is fairly far from our contemporary understanding of the science of arithmetic. On the other hand, it is very close to the ancient understanding of the importance of the teaching about numbers (i.e., arithmetic) as this art was constituted during the pre-Socratic period, especially in connection with the Pythagorean school.[1] Ancient and medieval scholars regarded numbers or numeric ratios as primarily representing the basic structure of reality, since they are the essences of things having a divine origin.[2] Arithmetic was regarded as not only the science allowing for conducting arithmetic operations (although this aspect – usually called *λογιστική* – was, as an applied practical arithmetic, part of the contemporary arithmetical art), but, above all, it was a scientific discipline with a significant philosophical and metaphysical overlap, since by the numbers cognition we at the same time cognize the metaphysical structure of reality.[3] Further, numbers are an instrument that sharpens the intellect and can move the human mind up from the erroneous world to the highest truths and the Divine essence.[4]

In the line starting with Pythagoras and early Pythagoreans, through Plato and Aristotle, over to some Neoplatonists and especially Neopythagoreans, the abovementioned importance of arithmetic, which is at the same time a methodical (or propaedeutic, at least) way to philosophical knowledge,[5] was handed down in different forms. During the Middle Ages, this understanding of mathematics was positively received, which may have been influenced by biblical texts suggesting that numbers were tools used by God in creating the world (most notably *Sap*. 11:20) and medieval mathematics often referred to this.[6]

It is not surprising, therefore, that the most influential arithmetic text of the Middle Ages was the loose Latin translation of the popular textbook by the Neopythagorean mathematician and philosopher Nicomachus of Gerasa called *Introduction to Arithmetic*,[7] contrived by Boethius around 500 AD.[8] In this translation, Boethius mediated to medieval intellectuals a summarization of the Neopythagorean teaching about numbers and about the importance of arithmetic itself.[9] According to this teaching, a number is not only an expression of quantity, but also a metaphysical entity we need to know in order to be able to grasp the world around us and to set out on our journey to God; numbers are patterns according to which God created all of creation and the arithmetic pursuit is at the same time an endeavour to grasp God's wisdom; no philosopher can be a true philosopher without devoting his time to studying mathematics – and especially arithmetic.[10]

Thus, arithmetic can be (not exclusively in medieval times) characterized as the science of numbers and, provided we want to understand it properly in the contemporary context, it is indispensable to focus on the ways number was defined. Nicomachus in his *Introduction to Arithmetic* lists three definitions of number:

(1) number as discrete quantity (*πλῆθος ὡρισμένον*);
(2) number as collection of units (*μονάδων σύστημα*);
(3) number as quantity emanating from unit (*ποσότητος χύμα ἐκ μονάδων συγκείμενον*).[11]

If judged by today's standards, Boethius's translation would not count as an illustrious piece of translation because (as the interpreter himself states in the dedication letter to his adoptive father Symmachus) it treats Nicomachus's text freely (*liberius*), some passages that seemed too extensive (*diffusius*) to Boethius were short-

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ened, while others were slightly extended when he thought that Nicomachus was too abrupt (*uelocius*).[12]

This approach to Nicomachus's arithmetic text is clearly reflected even in the case of the definition of number – in his translation, Boethius omits Nicomachus's first definition of number and mentions only the second and third definitions.[13] Since the definition of number is essential for grasping the content of arithmetic, Boethius's modification of the Greek original may seem rather surprising. Therefore, this paper follows two basic issues and tries to:

- elucidate possible reasons for Boethius's omission of one of the definitions of number given by Nicomachus,
- and at the same time, it focuses on all three definitions of number and shows that their formulation is directly connected to the problems which Nicomachean (and in relation to it even medieval) arithmetic tried to solve.

While pursuing these goals, I will define mathematics in accordance with Boethius's texts, per its relation to theoretical knowledge, and establish its place among the mathematical (so called quadrivial) sciences. Next, I will focus on arithmetic itself and introduce the areas of its interest through the various definitions of number as presented by Nicomachus. I will first describe number as discrete quantity (while this characterization is not included among Boethius's definitions for a certain reason, as I assume, this understanding of number holds a firm place in Boethius's account), then I will examine number as collection of units and, eventually, consider number as quantity emanating from unit.

2. Mathematics and the quadrivial sciences

In the abovementioned letter to Symmachus (i.e., the prologue to the translation of Nicomachus's *Introduction to Arithmetic*) Boethius characterizes arithmetic as the first of the four mathematical disciplines (*quattuor matheseos disciplinae*),[14] which are collectively referred to as the quadrivium (*quadruuium*), i.e., the four steps (*gradus*) leading up to philosophical wisdom.[15]

Boethius covered the exact place of mathematics in relation to philosophical knowledge in other texts. In the first commentary on Porphyry's *Isagoge* he endorsed the Aristotelian division of philosophy by splitting it into theoretical or speculative philosophy (*theoretica, speculatiua, contemplatiua*) and practical philosophy (*practica, actiua*). Within theoretical philosophy, he established three fundamental scientific domains concerned with *intellectibilia* (i.e., divine science), *intellegibilia* (i.e., mathematics) and *naturalia* (i.e., physics).[16] Similarly, in the later theological treatise *De trinitate* (*Quomodo trinitas unus Deus ac non tres dii*), Boethius divides theoretical philosophy into physics (*disciplina naturalis*), mathematics (*mathematica*) and theology (*disciplina theologica*). The subject of mathematics is defined as abstracted from matter and motion, even though it is present in matter as the forms (*formae*) of bodies.[17]

Mathematics is thus situated at the centre of theoretical philosophy.[18] Contrary to physics, it focuses on something stable, although not as metaphysically noble as theology (or metaphysics, i.e., the divine science) which inquiries into an object completely independent of matter.[19] At least since Aristotle, the subject of mathematics had been defined by the category of quantity which is removed from matter, unchanging and stable, but existing in the material world.[20] This description was accepted also by Nicomachus.[21]

Quantity (*ποσόν* or *ποσότης, quantitas*) can be divided into two basic kinds: on the one hand, it is a multitude (*πλῆθος, multitudo*), i.e., something firmly demarcated (*discreta*), delimited and countable, e.g. individual trees, books, etc.; and on the other hand, it is a magnitude (*μέγεθος, magnitudo*), i.e., something continuous (*continua*), with a certain extent and thus measurable, e.g. the length of an item, the circumference of a sphere, etc.[22] Boethius analysed the distinction between multitude and magnitude in detail and in a very similar manner in the passage about the category of quantity in his commentary on Aristotle's *Categories*.[23]

Cassiodorus, Boethius's contemporary and successor in the highest Roman office of *magister officiorum*, who knew Boethius's work including the translation of Nicomachus's *Arithmetic*,[24] thus defines the subject of mathematics in a link to Boethius as abstract quantity (*quantitas abstracta*), that is, quantity which is free from all delimitation, i.e., including the difference between countability and measurability.[25] Abstract quantity became the subject of general mathematical inquiry for medieval scholars, which was reinforced by the fact that Isidore of Seville (ca. 560–636) quoted Cassiodorus's Nicomachean-Boethian definition of the subject of mathematics literally in his encyclopaedia *Etymologies*, which was very popular in the Middle Ages.[26]

Boethius (and Nicomachus as well) used the multitude–magnitude distinction to distinguish between four special mathematical sciences, i.e., the *quadrivium*. Multitude can be thought of in itself (*per se*), that is, as discrete delimited multitude, i.e., number in itself, or as a multitude related to another multitude (*ad aliud, ad aliquid*), that is when numbers are ordered according to numerical ratios. The former gives rise to the doctrine of numbers, viz. arithmetic, which inquiries into numbers *per se*, the latter results in the science of music and musical intervals, whose subject are numerical ratios. The second kind of quantity, i.e., magnitude, can also be differentiated further. In this case, Boethius lists the criteria of stability (*immobilis*) and mobility (*mobilis*). The mathematical science that enquires into the unchanging and stable is geometry, while astronomy focuses on magnitudes in motion.[27] In this way, the basic structure of mathematics emerges as it was mediated through Boethius's Neopythagorean reading: the subject of arithmetic is *multitudo per se*, geometry focuses on *magnitudo stabilis*, music deals with *multitudo ad aliquid,* and astronomy is concerned with *magnitudo mobilis*.[28]

According to Boethius, arithmetic enjoys the most important position among the other mathematical disciplines (*principium et mater*), since multitude *per se* is nothing other than number itself, which is necessary for all other (not exclusively) mathematical sciences.[29] Without arithmetic, Boethius writes, there could be no geometry, music, astronomy, or any other kind of human knowledge at all. Boethius (following Nicomachus) confirms the primacy of arithmetic by the following argument: Numbers (*numerus*) are an expression of God's thoughts according to which God created all of creation, as mentioned before, therefore numbers must be antecedent (*prior*) by virtue of their nature (*natura*). When that which is later (*posterior*) vanishes, e.g. the species 'human' (*homo*), that which is antecedent, e.g. the genus 'animal' (*animal*) is not affected; while when that which is antecedent vanishes (animal), then all that is later and dependent on it (e.g. human) vanishes too.[30] Arithmetic as the science of numbers thus precedes all other sciences because nothing could exist without numbers.

Geometry, Boethius continues, needs arithmetic because it would not be able to think about the shapes (*formae*) of objects (e.g. triangle, quadrangle, etc.) without the ability to describe them using numbers. Music theory, i.e., numerical ratios (*proportiones*), would not be able to create various musical intervals (e.g. octave, perfect fourth, perfect fifth, etc.), if there were no numbers, and thus music needs arithmetic. Astronomy would lack the ability to describe the orbit of space bodies (*circuli, centra*, etc.) and their distances and positions without the knowledge of geometry (geometrical shapes) and music (perfect celestial harmony, *armonica*, music of spheres), therefore even in astronomy numbers are essentially present and without arithmetic there would be no geometry, music, and also no astronomy.[31]

In this manner, Boethius establishes a certain hierarchy of the mathematical sciences. Arithmetic is necessarily the first because for its purposes it needs to possess only numbers and nothing else is essential for it. Although geometry enquires into something *per se* (shapes), it needs numbers for its practice and follows immediately after arithmetic. Music does not focus on something *per se*, at the centre of its attention there are the relative properties of numbers, therefore it is also dependent on arithmetic (i.e., on numbers themselves) and comes third. In the case of astronomy, it is true that it necessarily needs both arithmetic and geometry and cannot operate even without music; therefore, the last fourth place among the mathematical sciences is due to it.[32]

3. *Quantitas discreta*: substance or definition of number?

All knowledge thus needs numbers and if humans want to pursue philosophy or science, they must start with numbers. Without numbers, there would be nothing, since everything is created of numbers or numerical ratios.[33] Boethius compares that to human speech when he says that in an analogous manner (i.e., according to a numerical order) syllables (*syllabae*) are created from letters (*litterae*) and then they proceed to fully articulated words (*voces*).[34] This example, e.g. the comparability of the relations between letters, syllables, and speech (*oratio*) and the relations between numbers and creation, is again clearly declared by Boethius in his commentary on Aristotle's *Categories*, where he also explicitly states that numbers are delimited quantities (*quantitas discreta*),[35] which could be understood as a basic characterisation of number itself.

Boethius's formulation bears a strong resemblance to the first Nicomachean definition of number as discrete quantity (*πλῆθος ὡρισμένον*), which Boethius did not include in his translation. There could be several reasons for Boethius's omission of this Nicomachus's definition.

The first possible reason could concern Boethius's knowledge of Nicomachus's treatise. For example, Boethius could have used the manuscript of the text that is missing this definition – e.g., the manuscript H (according to Hoche's apparatus) could be considered.[36] Further, provided that Boethius had encountered arithmetic in the circle of Ammonius Harmiae († ca. 520) in Alexandria, he would have adopted contemporary reading of Nicomachus's text as containing only two definitions of number, i.e. collection of units and "fluxion" theory.[37] An illustrative example of such interpretation is the commentary to Nicomachus's *Introduction to Arithmetic* written by Boethius's contemporary Asclepius of Tralles († ca. 555) where two definitions of number are mentioned in slightly confused form.[38]

On the other hand, (not only) older tradition of interpreting Nicomachus *Arithmetic* obviously preferred distinguishing three definitions of a number. For instance, Iamblichus, although his commentary to Nicomachus's *Introduction to Arithmetic* is very different in its scopes and aims, presupposed more definitions of number, including number as a discrete quantity (*πλῆθος ὡρισμένον*), i.e., Nicomachus's first definition of number.[39] Therefore, it may be fruitful to look at this definition in more detail.

Nicomachus's first definition of number is at least confusing, in part because of the ambiguous terminology used by Nicomachus in the introductory chapters of his arithmetic text.[40] While he first distinguished between two types of quantity (*ποσόν*) – multitude (*πλῆθος*) and magnitude (*μέγεθος*) –,[41] he subsequently used the term *ποσόν*, formerly declared as quantity, for differentiating between the two kinds of multitude which constitute the distinction between arithmetic and music. In contrast to multitude as (countable) quantity stands magnitude, characteristic for geometry and astronomy, for which Nicomachus in this case used the term *πηλίκος*.[42] Even this magnitude is obviously quantity (*ποσότης*), as stated by Nicomachus two chapters later.[43]

Nicomachus confusingly uses the terms quantity (*ποσόν, ποσότης*), multitude (*πλῆθος, ποσόν*) and magnitude (*μέγεθος, πηλίκος, ποσότης*). Numbers, provided they are defined as discrete quantity,[44] are by this definition primarily considered to be the subject of arithmetic because they must necessarily be multitude conceived in itself. Nicomachus is repeating what he has said earlier to a certain extent, since he specified the subject of arithmetic in a similar fashion a few lines above,[45] only using different terms.

Boethius apparently noticed the terminological ambiguity of Nicomachus's text[46] and tried to unify the involved terms when he used the phrase *multitudo per se*, where *multitudo* is a kind of quantity (*quantitas*) specified by delimitation (*discreta*). Therefore, he found it difficult to repeat Nicomachus's first definition of number as delimitated multitude, since he would *de facto* be saying that number is delimitation of delimitated quantity. For Boethius, number is not delimitated multitude (that is, something like *multitudo discreta*), but delimitated quantity itself (*quantitas discreta per se*). For Nicomachus and his ambiguous terminology this problem did not arise, since he at the same time confusingly referred to multitude as quantity.

However, terminological issues need not have been the essential reason (in addition to the abovementioned) why Boethius omitted Nicomachus's first definition. The redundancy and untrustworthiness of this definition may be due to the very effort to capture the substance (*substantia*) of number, which is the subject of Boethius's (and Nicomachus's) inquiry immediately prior to the formulations of number definitions. All things are formed according to numbers and numerical ratios and even these numbers are composed of certain components,[47] since numbers are not something entirely simple and are composed (*compositum*) by nature (*natura*). The components of a number cannot be diverse (*diversis*) but must manifest a certain similarity (*similis*). The substance of a number is composed of even (*par*) and odd (*impar*), which can be thought of as contradictory (*contraria*), but by composing even and odd it is possible to make up all the numbers that are used by God to create a harmonic composition (*modulatio*).[48]

The very substance of numbers is thus composite; therefore, it is discrete quantity since every composed substance is discrete and delimited (*discreta*) by its components.[49] If we speak about number as discrete quantity, then we do not define number but describe its substance and, at the same time, we can characterize the place of arithmetic among the mathematical sciences in the same manner (mathematics enquires into abstract quantity, arithmetic deals with discrete quantity).[50]

The distinction between a definition and a description is mentioned by Boethius at several paragraphs of his treatises. According to *Commentary to Categories* definitions (*diffinitiones*) and descriptions (*descriptiones*) are related to the nature or notion of things (*ratio substantiae*) – a definition must be composed from a superior genus (*genus*) and a difference (*differentia*), whereas a description collects the properties (*proprietates*) of certain thing (*res*).[51] Accordingly, a definition must express what the given thing is (*quid sit*).[52] In other words, according to Boethius's second commentary to Porphyry's *Isagoge*: a definition (*definitio*) shows a common (*communia*) substance of many (*multa*) things, while a description (*descriptio*) expresses specific properties (*proprietates*) or qualities of the given thing. If we know properties, we can use a description, but if we know the genus and specific difference, we can formulate a definition, ergo we know a nature or an essence of the thing.[53]

According to these Boethius's statements, a definition must express what the given thing is, it means nature, essence, or substance of the thing. It is beyond doubt that he adhered to the same idea of definition when he was translating the arithmetical textbook by Nicomachus, since he makes an identical statement even in the *Introduction to Arithmetic*: *quid sit numerus definiendum est*;[54] and he immediately adds the first division (*divisio*) of numbers which is nothing else than division to even and odd numbers, i.e., definition of the basic components of which the substance of numbers is composed.[55]

The delimitation of even and odd (or similar delimitations) belongs to the substance of numbers, but through the components of number the real definition of number is not expressed – there is no superior genus or specific difference. The components of a number (even or odd) are the basic properties of a number, so we should understand it as a description of a number. Characterizing number as discrete quantity is not a proper definition of number, but merely an expression of the fact that a number is always composed of certain components, i.e., it describes the substance of numbers. That means: any number is always a certain quantity delimited by that particular quantity (e.g. even or odd, specific value of the number, etc.), but this is not a definition.

Therefore, Boethius would have denied the status of proper definition of number in this case, because it is not an answer to the question what number is, but to the question of what the substance of number is composed, while it only defines these components afterwards. This may have been the key reason for Boethius to omit this first definition of number.[56]

Nonetheless, Boethius did understand number as discrete quantity (although discrete quantity cannot be a definition of number) – it is something that belongs to the very substance of numbers, because each number must express some discrete quantity.[57] Arithmetic deals with numbers and its goal is to appreciate the properties of numbers, that is, to discern (*divisio*) the characteristics of number, i.e., discrete quantity *per se*, as described by Boethius himself.[58] The first and main topic of Boethius's *Introduction to Arithmetic* is to specify the various properties of numbers, provided we start from the division to even and odd.

Thus, like Nicomachus in the text Boethius was working with, Boethius gradually deals with the properties of number *per se*, introduces the definitions of even and odd, clarifies the reasons why *unitas* is not a number, but the source, cause and mother of all numbers,[59] and then classifies numbers to explain their properties according to different criteria. Even numbers are further divided to even times even (*pariter par*), even times odd (*pariter impar*), odd times even (*impariter par*), and their arithmetic properties are described. With respect to odd numbers, Boethius discerns prime numbers, i.e., primary and incomposite numbers (*primi et incompositi*), secondary and composite numbers (*secondi et compositi*), and numbers he calls middle (*medii*). He also addresses their properties and instructions for determining sequences of those types of numbers, among others he illustratively describes the so-called sieve of Eratosthenes (*Eratosthenes cribrum*), which serves to identify all prime numbers. By the end of this first thematic part of his textbook, Boethius returns to even numbers and introduces another type of division according to the criterion of the sum of their dividers which results in an integral number (quotient), therefore he discerns perfect numbers (*perfecti*), superfluous numbers (*superflui*) and diminutive numbers (*deminuti*). The algorithm for finding perfect numbers is again clearly described together with the properties of these types of numbers. The overview of the arithmetic teaching about the properties of numbers *per se* constitutes the larger part of the first book in Boethius's translation.[60]

It seems clear that the substance of number, which is composed of even and odd and which Nicomachus used in his first definition of number, is the main and most important content of the arithmetic science. Number as discrete quantity in itself includes a delimitation of properties (i.e., a description) according to the classificatory criteria which were used by ancient and medieval arithmetic and was its first and most significant topic.

4. Number as *collectio unitatum*

However, arithmetic does not deal only with number, i.e., discrete quantity in itself, that is multitude in itself, but, as mentioned above, it is the science without which the other mathematical sciences would not be possible. This seems to affect the definition of the subject of arithmetic, that is, number. The first definition mentioned by Boethius (the second one mentioned by Nicomachus) was the most widely used delimitation of number in (not only) the Middle Ages.[61] Numbers are defined as collections of units (*unitatum collectio*).[62] In slight variations, this definition is mentioned by all scholars at the turn of antiquity and the Middle Ages whose texts were read at the institutions of medieval education: Martianus Capella wrote about 'congregation of units' (*congregatio monadum*),[63] Cassiodorus defined number as multitude composed of units (*ex monadibus multitudo composita*),[64] Isidore of Seville defined it as multitude built up of units (*multitudo ex unitatibus constituta*).[65]

This definition of number clearly refers to the above-mentioned difference between unit and a number: a number is something composed of units, while unit is not a number but the source, root, cause and mother of numbers. The lowest number is thus number two, since it is the first real collection of units.[66] At the same time, it is clear that by this definition Boethius (and Nicomachus) followed one of the essential themes of the ancient Pythagorean tradition of arithmetic, i.e., figurate numbers.[67]

Figurate numbers are a certain form of transition between arithmetic and geometric teachings since these numbers express the areal or spatial representation of numerical values with the help of unit points ordered into geometrical shapes. This was explicitly highlighted by Boethius when he stated at the beginning of the passage about figurate numbers in his translation of Nicomachus's work that he was going to enquire into quantity, which is not related to something else, but stands by itself and relates to geometric figures. He recalls that geometry as a science originates in arithmetic and uses this as evidence for the importance of these issues for arithmetic because it leads to the next one of the mathematical sciences.[68]

Defining number as collection of units gives rise to the image of a figural representation of numbers. The ordering of units, or representing them with points (*puncti*) as geometrical shapes, corresponds to natural (*naturalis*) signs for numerical values, in contrast to Greek (Nicomachus) or Roman (Boethius) numerals, and we could add also to Arabic numerals. While these commonly used numerical symbols (*signa numerorum*) are instituted by humans, figurate numbers show numbers as sets of units, that is, if the number 5 is expressed, then this numeral symbol does not correspond to the natural character of the value '5', as this can be achieved only by an ordering of five units together.[69]

The insight that numbers are collections of units ordered into certain geometrical shapes reveals the direct relation of numbers to the created world surrounding human beings, whose building blocks are geometrical forms. When units are ordered into lines, we get linear numbers (*numeri lineares*) characterized by longitude (*longitudo*) as the only direction or dimension (*interuallum*). When the points are ordered in two directions (besides longitude there is also latitude, *latitudo*), plane numbers arise (*numeri plani*), e.g. triangle numbers (*triangulares*), tetragonal numbers (*quadrati*), pentagonal numbers (*pentagoni*), etc. When a third dimension, altitude (*altitude*), is added to length and width, we get solid numbers (*numeri solidi*), that is, pyramidal numbers (*pyramides*), cubic numbers (*cybi*), etc. These types of numbers are similarly characterized by Boethius and Nicomachus, especially with respect to the way of establishing how figurate numbers refer to various numerical ratios.[70] The relationship to geometry is obvious here, therefore it is not surprising that number was defined in this way by Euclid in the arithmetic book of his *Elements*.[71]

The largest part of the second book of Boethius's *Introduction to Arithmetic* is constituted by a treatise on figurate numbers, which fully corresponds to the definition of number as *collectio unitatum*, while Boethius's own words clearly show that in the case of figurate numbers we are confronted with nothing else than orderings of units. This definition seems to directly refer to the second big topic of the arithmetic of his time. While the description of number as discrete quantity immediately leads to the explanation of the properties of numbers which are constitutive of numbers (i.e., even and odd) and as such essentially belongs to arithmetic, the definition of number as ordered set of units leads to what number is (i.e., definition of number: collection represents the superior genus and units as a specific difference forms each number) and also to the recognition of how important number is – in this case primarily in geometry as a mathematical science.

5. Number as stream issuing from unit (and returning back to it again)

The second definition of number, as listed by Boethius (and the last one listed by Nicomachus), is number as big mass of quantity issuing from unities (*quantitatis aceruus ex unitatibus profusus*).[72] Even in this case, number is fundamentally differentiated from unit since it grows out of unit and unit is once again the cause of all numbers, not a number. Contrary to the definition of number as collection of units, the role of unit is now different: it is the source of numbers, while numbers are specific streams which come out of one, with which they are in a certain proportional relation.

Among the texts from the turn of antiquity and the Middle Ages, Martianus Capella stressed this definition strongly. In his allegoric text, arithmetic (as well as the other liberal arts) is depicted as a lady presenting her craft. The description of the arithmetical science and its personified appearance reflect the understanding of numbers as streams. From the forehead (*frons*) of Lady Arithmetic a barely visible mystical ray (*radius*) bursts out (*erumpens*), which branches out and grows until it starts to shrink and contract (*contrahens*) back to her forehead.[73] The ray probably represents numbers and sequences, which branch out, according to specific ratios, into mutually interconnected, yet individual types of numbers ordered per these ratios which trace their origin back to unit. These numbers can be retroactively converted to the cause of all numbers, i.e., unit. With this image, Martianus was perhaps presupposing this definition of number as stream of multitude, which comes out of the primary source and (as noted by Capella) returns to it (*a monade veniens multitudo atque in monadem desinens*).[74]

Boethius (possibly inspired by Nicomachus's text) used the term 'quantity' (*quantitas*) in his definition, by which he probably referred to multitude (*multitudo*), as it is the case in Martianus's work. Even in this case, Boethius meant discrete quantity, although this time it was *ad aliquid*, not *per se*, that is, numbers with respect to their relative properties. When we understand numbers as streams coming out of (divine) unit and creating sequences (per specific ratios) that are based on the relations (ratios) between the numbers of this series, in compliance with certain rules, we subscribe to the view of numbers as relative to other numbers allowing to establish proportional relations.[75]

These relative properties of numbers are, per definition, immediately necessary for music, which enquires into multitude related to something else, i.e., ratios.[76] From the point of view of arithmetic, numbers as numerical ratios are therefore divided into two main genera – equality (*aequalitas*) and inequality (*inaequalitas*). While the former involves two numerical values that do not differ (e.g. 5 and 5), in the latter we are dealing with numbers expressing different quantitative values (e.g. 5 and 10). While there is only one kind of equality, there are several kinds of inequality.[77] The basic division of inequalities distinguishes between numbers in which a higher number is compared to a lower one (they are derived from multiples, *multiplex*) and another kind of numbers in which a lower number is compared to a higher one (derived from dividers, *submultiplex*). Both types of inequalities include five species of dissimilar relations between numbers. In the first case, in which a higher number is compared to a lower one, Boethius distinguishes between multiples (*multiplex*), superparticular numbers (*superparticularis*), superpartient numbers (*superpartiens*), superparticular multiples (*multiplex superparticularis*) and superpartient multiples (*multiplex superpartiens*). In a similar way, he distinguishes between five species of numbers when a lower number is compared to a higher one, i.e., submultiplex, subsuperparticular, subsuperpartient, submultiplex subsuperparticular and submultiplex subsuperpartient.

All types of inequality originate from equality and Boethius lists three simple rules[78] which can be applied to original equality (ratio 1 : 1) in order to get multiples (2 : 1, then 3 : 1, etc.), from multiples superparicular numbers (2 : 1 → 3 : 2; 3 : 1 → 4 : 3 etc.) are obtained; from superparticular numbers either superpartient numbers (3 : 2 → 5 : 3; 4 : 3 → 7 : 4 etc.) or superparticular multiples (3 : 2 → 5 : 2; 4 : 3 → 7 : 4 etc.) are derived, and from superpartient numbers superpartient multiples (5 : 3 → 8 : 3; 7 : 4 → 11 : 4 etc.) are formed. These kinds of inequality are apparently nothing else than streams coming out of unit and creating relative multitudes. Boethius describes in detail not only the origin of these types of numbers; he also provides elaborate characteristics of them and describes their mutual relations (especially between multiples and superparticular numbers), which can be widely used also in other sciences and in particular in music.[79]

One passage of Boethius's considerations about the relative properties of numbers is worth mentioning in order to highlight the connection between these issues and the definition of number. While defining number, Boethius only says that a number is a mass of quantity which comes out of unit, although it seems obvious that he, like Martianus Capella, considered the possible return of these inequalities back to the original equality (unit). At the beginning of the second book of *Introduction to Arithmetic* he introduces a second set of simple rules,[80] which can serve as an easy-to-use tool for converting three-member sequences (i.e., some kind of inequality) to what these inequalities originated from. It is clear that this second set of three rules is reversible to the first set of rules. In this way, all inequalities can be converted to earlier inequalities and in the last step they can be reduced to equality,[81] which fully corresponds to the broader version of this definition of number in Martianus's work.

The overview of the relative properties of numbers in the *Introduction to Arithmetic* is interrupted by the abovementioned passage about figurate numbers, which are revisited by Boethius at the end of his work where he deals with numerical sequences (*proportionalitates*). While figurate numbers prepare ground for geometric science and the relative properties of numbers do the same for music, numerical sequences serve in a certain way as an introduction to astronomy, besides the other mathematical sciences. Boethius first describes sequences generally (i.e., series of – at least – three numbers, whose values are given according to given rules of numerical ratios) and then he introduces ten different types. The main focus is on three of them: (1) arithmetic, where the relation between numbers is given by the difference; (2) geometric, where the values of numbers are delimited by numerical ratios, i.e., quotients; and (3) harmonic sequences, which originate from three numbers in cases when the ratio of the third and first member of this sequence is equal to the ratio between the difference of the third and second member and the difference between the second and first member of this sequence. Specific instructions for identifying the middles of these sequences are also included and further relations between the sequences and figurate numbers are revealed with references to the arithmetic cohesion of geometry and music.[82]

In the last chapter of *Introduction to Arithmetic*, Boethius (and Nicomachus) pays attention to the highest and most perfect harmony (*maxima perfectaque armonia*), which in a four-member succession encompasses musical intervals and, at the same time, represents the ordering of nature (*natura*), i.e., issues relevant for the ordering of the universe (astronomy and geometry). Perfect harmony captures all three listed sequences by using four numbers (the sequence of the numbers 6, 8, 9, 12 is cited as an example).[83]

The inquiry into numerical ratios and numerical sequences fully corresponds to the definition of numbers as streams or big mass of quantity (i.e. genus) coming out of unit and returning back to it again (i.e., specific difference). It is thus possible to say that, like the previous definition of number and description of the substance of number, the last definition immediately characterizes these big topics of Boethian-Nicomachean arithmetic.

6. Conclusion

As implied above, the content of Boethius's *Introduction to Arithmetic* can be divided into several basic thematic areas. After (0) the introductory delimitation of arithmetic, its importance, subject and goal, including its relations to other mathematical sciences (I, 1–3), it deals with four fundamental issues:

(1) properties of numbers as themselves (I, 3–20);
(2) properties of numbers in relation to other numbers, i.e., numerical ratios (I, 21–II, 3);
(3) figurate numbers (II, 4–39);
(4) numerical sequences (II, 40–54).

Everything is in accordance with Nicomachus's original text, although the translation differs from that in several details.

The ordering of the subject matter is firmly structured. While the properties of number in itself (ad 1) are essentially related to the subject of arithmetic because they describe *quantitas discreta per se*, i.e., multitude in itself, the relative properties of numbers also include a certain overlap with music, since they characterize *quantitas discreta ad aliud*, i.e., a multitude related to another multitude. After multitude is dealt with, attention is focused on numbers which are applied to descriptions of magnitude (*quantitas continua*) – therefore, numbers as the basis for geometrical knowledge are introduced (figurate numbers) and, eventually, the way how numbers are reflected in all the mathematical sciences, that is, among others, how they can create the perfect harmony which is to be found in the cosmos. Boethius acknowledges this thematic transition explicitly[84] and at the same time he fully keeps his division of the mathematical science from the introductory chapters of his translation and his declared intention from the foreword (the accompanying letter), i.e., the endeavour to show why and in what manner arithmetic is the first of the mathematical sciences.

Boethius's definitions of number and the way they differ from those of Nicomachus also seem to be leading (in accordance with the structure of the whole text) to the same goals. When Boethius characterizes number as discrete quantity in itself, he does not include it among the definitions of number, which could be due to several abovementioned reasons, including Nicomachus's not very transparent terminology. Boethius tries to use terms consistently but, above all, he suggests that when number is delimitated in this way, it is not a true definition but rather a description of the components of its substance. By this he also sets down the characterization of number for the first big topic of arithmetic (ad 1), i.e., delimitating the properties of numbers *per se* according to various criteria. The definition of number as collection of units directly related to the mathematical science dealing with something that is *per se*, i.e., geometry. This definition actually says what number is and, on top of that, it shows the importance of number for other mathematical sciences, which fully corresponds to the third (ad 3) domain of arithmetic subject matter, i.e., figurate numbers. The last definition of number, which speaks of big mass of quantity coming out of unit (and eventually coming back to it), refers to the properties of numbers that are not *per se* but related to something else. This is typical for music (and partially even for the movements examined in astronomy) and corresponds to the second (ad 2) and fourth (ad 4) key topic of arithmetic subject matter (in the case of the last thematic transition, Boethius himself mentions that he is returning to an interrupted topic, therefore he was fully aware of the direct interconnectedness between the two issues), since the very idea of stream refers to the mutual cohesion and versatility of number apprehended in this way.

Nicomachus's three definitions are thus at the same time references to the inherent topics of arithmetic as understood by the Boethian-Nicomachean tradition of this science. These three topics was crucial for (not only) early medieval arithmetic, as was briefly documented in this paper. Boethius, as this paper has tried to show, found the first definition inappropriate, but he had to consider it (on conceptual and factual grounds) as a fundamental description of the substance of number, hence he included it in a modified version (not as a definition) in his arithmetic work as well.

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Notes

[1] Cf. M. Kline, *Mathematical Thought from Ancient to Modern Times*. Vol. 1. New York – Oxford: Oxford University Press, 1972, p. 29, or C. Rowett, "Philosophy's Numerical Turn. Why the Pythagoreans's Interest in Numbers is Truly Awesome." In: D. Sider & D. Obbink (Eds.), *Doctrine and Doxography. Studies on Heraclitus and Pythagoras*. Berlin, Boston: De Gruyter, 2013, pp. 3–31.

[2] Cf. R. Netz, "The Pythagoreans." In: T. Koetsier & L. Bergmans (Eds.), *Mathematics and the Divine: A Historical Study*. Amsterdam et al.: Springer, 2005, pp. 91–96, or I. Mueller, "Mathematics and the Divine in Plato." In: T. Koetsier & L. Bergmans (Eds.), *Mathematics and the Divine*, pp. 101–121.

[3] See, e.g., D. Albertson, *Mathematical Theologies. Nicholas of Cusa and the Legacy of Thierry of Chartres*. Oxford: Oxford University Press, 2014, pp. 23–39, H. Chadwick, *Boethius. The Consolations of Music, Logic, Theology, and Philosophy*. Oxford: Clarendon Press, 1981, pp. 71–74, or M. Kline, *Mathematics and the Search for Knowledge*. New York – Oxford: Oxford University Press, 1986, pp. 39–43.

[4] Cf. Plato, *Resp.* VII, 8, 524d–526c.

[5] See, e.g., J. Caldwell, "The *De Institutione Arithmetica* and the *De Institutione Musica*." In: M. Gibson (Ed.), *Boethius. His Life, Thought and Influence*. Oxford: Basil Blackwell, 1981, pp. 135–137, or W. Bernard, "Zur Begründung der mathematischen Wissenschaften bei Boethius." *Antike und Abendland* 43 (1997), pp. 65–69.

[6] For patristic reading, see, e.g., I. Peri, "'*Omnia mensura et numero et pondere disposuisti*': Die Auslegung von Weish. 11,20 in der lateinischen Patristik." In: A. Zimmermann (Ed.), *Mensura: Mass, Zahl, Zahlensymbolik im Mittelalter*. Bd. 1. Berlin: De Gryuter, 1983, pp. 1–21, cf. early medieval text – *De arith. Boeth.*, ad prol., p. 128,53–57 or Abbo of Fleury, *In Vict. Calc.* II, 9–11, p. 69; II, 15, p. 71.

[7] For a delimitation of the importance of arithmetic in Nicomachus, see D. O'Meara, *Pythagoras Revived. Mathematics and Philosophy in Late Antiquity*. Oxford: Clarendon Press, 1997, pp. 14–23, or J.-F. Mattéi, "Nicomachus of Gerasa and the Arithmetic Scale of the Divine." In: T. Koetsier & L. Bergmans (Eds.), *Mathematics and the Divine*, pp. 125–132.

[8] For dating, see J.-Y. Guillaumin, "Boethius's *De institutione arithmetica* and its Influence on Posterity." In: N. H. Kaylor & P. E. Phillips (Eds.), *A Companion to Boethius in the Middle Ages*. Leiden: Brill, 2012, p. 135, or P. E. Phillips, "Anicius Manlius Severinus Boethius: A Chronology and Selected Annotated Bibliography." In: N. H. Kaylor, Jr. & P. E. Phillips (Eds.), *A Companion to Boethius in the Middle Ages,* p. 552.

[9] Cf. M. Masi, "Boethius' *De institutione arithmetica* in the Context of Medieval Mathematics." In: L. Obertello (Ed.), *Atti del Congresso internazionale di studi Boeziani*. Roma: Herder, 1981, pp. 263–272, P. Kibre, "The Boethian *De Institutione Arithmetica* and the Quadrivium in the Thirteenth Century University Milieu at Paris." In: M. Masi (Ed.), *Boethius and the Liberal Arts*. Bern: Peter Lang, 1981, pp. 67–80, A. White, "Boethius in the Medieval Quadrivium." In: M. Gibson (Ed.), *Boethius*, pp. 162–205, or I. Caiazzo, "Medieval Commentaries on Boethius's *De arithmetica*: A Provisional Handlist." *Bulletin de philosophie médiévale* 62 (2020), pp. 3–4.

[10] Boethius, *Arith.* I, 1–2, pp. 9,5–15,23; cf. A. Kijewska, "Mathematics as a Preparation for Theology: Boethius, Eriugena, Thierry of Chartres." In: A. Gallonier (Ed.), *Boèce ou la chaîne des saviors*. Louvain – Paris: Éditions Peeters, 2003, pp. 635–636.
[11] Nicomachus, *Intr. arith.* I, 7, 3, p. 13,7–8. English translation by M. L. D'Ooge (Nicomachus of Gerasa, *Introduction to Arithmetic*. Transl. M. L. D'Ooge. New York – London: Macmillan, 1926, p. 190): "Number is limited multitude or a combination of units or a flow of quantity made up of units…" For context, see, e.g., F. M. Petrucci, "Theon of Smyrna: Re-thinking Platonic Mathematics in Middle Platonism." In: H. Tarrant et al. (Eds.), *Brill's Companion to the Reception of Plato in Antiquity*. Leiden – Boston: Brill, 2018, pp. 144–146.
[12] Boethius, *Arith.*, prol., p. 5,44–53. There are numerous texts analysing Boethius's understanding of translation (method, aims, form, etc.) – for instance, C. Vogel, *Boethius' Übersetzungsprojekt. Philosophische Grundlagen und didaktische Methoden eines spätantiken Wissenstransfers*. Wiesbaden: O. Harrasowitz, 2016, especially pp. 125–169, A. Kárpáti, "Translation or Compilation? Contributions to the Analysis of Sources of Boethius' *De institutione musica*." *Studia Musicologica Academiae Scientiarum Hungariae* 29 (1987), pp. 5–33, C. Bower, "Boethius and Nichomachus: An Essay Concerning the Sources of *De institutione musica*." *Vivarium* 16/1 (1978), pp. 1–45), C. Vogel, "Die „boethianische Frage" – Über die Eigenständigkeit von Boethius' logischem Lehrwerk." *Working Paper des SFB 980 Episteme in Bewegung* 17 (2019), pp. 23–24, J. Barnes, "Boethius and the Study of Logic." In: M. Gibson (Ed.), *Boethius* p. 79), A. E. Moyer, "The *Quadrivium* and the Decline of Boethian Influence." In: N. H. Kaylor, Jr. & P. E. Phillips (Eds.), *A Companion to Boethius in the Middle Ages*, p. 481), etc.
[13] Boethius, *Arith.*, I, 3, pp. 15,2–16,4. English translation by M. Masi (M. Masi, *Boethian Number Theory. A Translation of the* De Institutione Arithmetica. Amsterdam: Rodopi B.V., 1983, p. 114): "A number is a collection of unities, or a big mass of quantity issuing from unities."
[14] Boethius, *Arith.*, prol., p. 5,54–55; cf. M. Fournier, "Boethius and the Consolation of the Quadrivium." *Medievalia et Humanistica. Studies in Medieval and Renaissance Culture* 34 (2008), pp. 3–4.
[15] Boethius, *Arith.* I, 1, p. 9,5–7, or p. 11,64–72.
[16] Boethius, *1 In Isag.* I, 3, pp. 8,1–9,8; cf. A. Hicks, *Composing the World. Harmony in the Medieval Platonic Cosmos*. Oxford: Oxford University Press, 2017, pp. 70–73.
[17] Boethius, *De Trin.* II, p. 8,5–16.
[18] For further reading, see H. M. Klinkenberg, "Divisio philosophiae." In: I. Craemer-Ruegenberg & A. Speer (Eds.), *Scientia und ars im Hoch- und Spätmittelalter*. Bd. I. Berlin: De Gruyter, 1994, pp. 3–19, J. A. Weisheipl, "The Concept of Scientific Knowledge in Greek Philosophy." In: A. Gagne & T. De Koninck (Eds.), *Melanges a la Memoire de Charles De Koninck*. Quebec: Presses de l'Universitè Laval, 1968, pp. 487–507, or idem, "The Nature, Scope, and Classification of the Sciences." In: D. C. Lindberg (Ed.), *Science in the Middle Ages*. Chicago: University of Chicago Press, 1977, pp. 461–482.
[19] Cf. Aristotle, *Met.* XI, 7, 1064a–b.
[20] See e.g. Aristotle, *De an.* III, 7, 431b, *Phys.* II, 2, 193b–194a, or *Met.* VI, 1, 1026a and many others. For more details, see e,g, P. Studtmann, "Aristotle's Category of Quantity: A Unified Interpretation." *Apeiron* 37/1 (2004), pp. 69–91.
[21] Nicomachus, *Intr. arith.* I, 2, 5, p. 5,10–12.
[22] Ibid. I, 2, 4–5, pp. 4,13–5,10, Boethius, *Arith.* I, 1, p. 10,23–30; cf. L. Corti, "Scepticism, number and appearances. The ἀριθμητικὴ τέχνη and Sextus' targets in M I-VI." *Philosophie antique* 15 (2015), pp. 129–130.
[23] Boethius, *In Cat.* II, c. 201C–205B; cf. A. Hicks, "Martianus Capella and the Liberal Arts." In: R. Hexter & D. Townsend (Eds.), *The Oxford Handbook of Medieval Latin Literature*. Oxford: Oxford University Press, 2012, pp. 312–313. For dating, see L. M. De Rijk, "On the Chronology of Boethius' works on logic II." *Vivarium* 2 (1964), p. 125.
[24] Cassiodorus, *Var.* I, 45, 4, p. 40,12, or Cassiodorus, *Inst.* II, 4, 7, p. 140,17–19; cf. J. Caldwell, "The *De Institutione Arithmetica* and the *De Institutione Musica*." In: M. Gibson (Ed.), *Boethius*, pp. 137–138.
[25] Cassiodorus, *Inst.* II, 3, 21, p. 130,18–22; cf. J.-Y. Guillaumin, "Boethius's *De institutione arithmetica*...", In: N. H. Kaylor & P. E. Phillips (Eds.), *A Companion to Boethius in the Middle Ages*, pp. 153–155.
[26] Isidore, *Etym*. III, preaf., l. 1–5. For more details of the influence of this book, see, for example, A. Fear & J. Wood (Eds.), *Isidore of Seville and his Reception in the Early Middle Ages. Transmitting and Transforming Knowledge*. Amsterdam: Amsterdam University Press, 2016, e.g., S. O'Sullivan, "Isidore in the Carolingian and Ottonian Worlds: Encyclopaedism and Etymology, c. 800–1050. In: A. Fear & J. Wood (Eds.), *A Companion to Isidore of Seville*, pp. 524–568.
[27] Boethius, *Arith.* I, 1, pp. 10,39–11,43, Nicomachus, *Intr. arith.* I, 3, 1–2, pp. 5,13–6,7; cf. G. R. Evans, "Boethius' Geometry and the Four Ways." *Centaurus* 25/2 (1981), pp. 161–165.
[28] Cf., e.g., U. Pizzani, "Il *Quadrivium* Boeziano e i suoi problem." In: L. Obertello (Ed.), *Atti del Congresso internazionale di studi Boeziani*. Roma: Herder, 1981, pp. 211–226, M. Masi, "The Liberal Arts and Gerardus Ruffus' Commentary on the Boethian *De Arithemtica*." *Sixteenth Century Journal* 10/2 (1979), p. 29, or U. Pizzani, "Studi sulle fonti del «De Institutione Musica» di Boezio." *Sacris erudiri* 16 (1965), p. 158 etc. For the reception of the quadrivium division in Middle Ages with illustrative diagrams, see D. K. Walden, "Charting Boethius: Music and the Diagrammatic Tree in the Cambridge University Library's *De Institutione Arithmetica*, MS II.3.12." *Early Music History* 34 (2015), pp. 207–228.
[29] For early medieval adaptation, see G. R. Evans, "Introductions to Boethius's "Arithmetica" of the Tenth to the Fourteenth Century." *History of Science* 16 (1978), p. 36 or *De arith. Boeth.*, ad I,I, pp. 132,171–133,188.
[30] Boethius, *Arith.* I, 1, p. 12,73–90, Nicomachus, *Intr. arith.* I, 4, 1–3, pp. 9,5–10,9; cf. M. Fournier, "Boethius and the Consolation of the Quadrivium,", p. 3 and *De arith. Boeth.*, ad I,I, p. 133,200–210.
[31] Boethius, *Arith.* I, 1, pp. 12,91–14,130, Nicomachus, *Intr. arith.* I, 4, 4–5, 3, pp. 10,9–11,23; cf. J.-Y. Guillaumin, "Boethius's *De institutione arithmetica*…," pp. 139–141.
[32] During late antiquity and in the Middle Ages, other methods of the ordering four mathematical sciences existed. E.g. Claudius Ptolemy understood astronomy as the most important science not only among the mathematical sciences; he described it as a more fundamental discipline for human knowledge than metaphysics or theology, that is, in some respects, as the most valuable of all the components of theoretical philosophy – see Ptolemy, *Alm.* I, 1, pp. 5,7–7,24. Another example of a different ordering of the mathematical sciences of the quadrivium could be Boethius's predecessor, the encyclopaedist Martianus Capella who in his – widely studied, commented, and quoted – cf. M. Teeuwen, "The Pursuit of Secular Learning: The Oldest Commentary Tradition on Martianus Capella." *Journal of Medieval Latin* 18 (2008), pp. 36–51, or W. H. Stahl, *The Quadrivium of Martianus Capella: Latin Tradition in Mathematical Sciences 50 B.C.–A.D. 1250. Martianus Capella and the Seven Liberal Arts*. Vol. 1. New York – London: Columbia University Press, 1971 – work *On the Marriage of Philology and Mercury* places geometry in the first place and only then comes arithmetic, followed by astronomy and music – see Martianus, *De nupt.* VI, 567–IX, 996, pp. 201,7–384,17. Other similar examples could be listed.
[33] Boethius, *Arith.* I, 2, p. 14,2–6, Nicomachus, *Intr. arith.* I, 6, 1, p. 12,1–12; cf. G. R. Evans, "Introductions to Boethius's "Arithmetica"…," p. 36.
[34] Boethius, *Arith.* II, 1, p. 93,7–9, Nicomachus, *Intr. arith.* II, 1, 1, p. 73,7–10; cf. *De arith. Boeth.*, ad II,I, p. 143,465–469.
[35] Boethius, *In Cat.* II, c. 203B, cf. M. Asztalos, "Nomen and Vocabulum in Boethius's Theory of Predication." In: T. Böhm & T. Jürgasch & A. Kirchner (Eds.), *Boethius as a Paradigm of Late Ancient Thought*. Berlin – Boston: De Gruyter, 2014, pp. 35–39.
[36] See Nicomachus, *Intr. arith.* I, 7, 1, p. 13 (n. ad VII.7); cf. Boèce, *Institution Arithmétique*. Ed. & transl. J.-Y. Guillaumin. Paris: Les Belles Lettres, 1995, p. 186, n. 44.
[37] Cf. J.-Y. Guillaumin, "La structure du chapitre 1, 4 de l'*Institution Arithmétique* de Boèce et le cours d'Ammonios sur Nicomaque." *Revue d'histoire des sciences* 47/2 (1994), pp. 253–257, or L. Tarán, "Asclepius of Tralles: Commentary to Nicomachus' *Introduction to Artihmetic*." *Transactions of the American Philosophical Society* 59/4 (1969), p. 77.
[38] Asclepius of Tralles, *In Nicom. Arith.* I, μθ, p. 32,1–3.
[39] Iamblichus, *In Nicom. Arith*., p. 10,18; see L. Tarán, "Asclepius of Tralles…", pp. 15–17, or G. Radke, *Die Theorie der Zahl im Platonismus: Ein systematisches Lehrbuch*. Tübingen: A. Francke, 2003, pp. 226–227.
[40] Cf. in similar context J.-Y. Guillaumin, "La structure du chapitre…,“ pp. 250–252.
[41] Nicomachus, *Intr. arith.* I, 2, 4–5, pp. 4,13–5,10.
[42] Ibid. I, 3, 1–2, pp. 5,13–6,7.
[43] Ibid. I, 5, 2, p. 11,16–18.
[44] Ibid. I, 7, 1, p. 13,7.
[45] Ibid. I, 3, 1, p. 5,13–14.
[46] Cf. J.-Y. Guillaumin, "La structure du chapitre…,“ pp. 251–253.
[47] Cf. C. Vogel, C. Vogel, *Boethius' Übersetzungsprojekt*…, pp. 25–27, S. C. McCluskey, "Boethius's Astronomy and Cosmology." In: N. H. Kaylor, Jr. & P. E. Phillips (Eds.), *A Companion to Boethius*, pp. 47–73, here pp. 54–55, or W. Bernard, "Zur Begründung der mathematischen Wissenschaften…," pp. 69–70.

[48] Boethius, *Arith.* I, 2, pp. 14,6–15,13, Nicomachus, *Intr. arith.* I, 6, 2–4, pp. 12,12–13,6; cf. *De arith. Boeth.*, ad I,II, p. 136,260–272.
[49] Boethius, *Arith.* I, 2, p. 15,12–14; cf. *In Boeth. Arith*. 14, p. 251.
[50] Cf. Cassiodorus, *Inst.* II, 4, 2, p. 133,8.
[51] Boethius, *In Cat.* I, c. 166A.
[52] Ibid., c. 163D. Cf. also Boethius, *In Cic. Top*. III, p. 318,35–37.
[53] Boethius, *2 In Isag*. I, 7, pp. 153,9–154,8; cf. J. Magee, *Boethius on Signification and Mind*. Leiden – New York et al.: Brill, 1989, pp. 123–124.
[54] Boethius, *Arith.* I, 3, p. 15,1.
[55] Ibid. I, 3, p. 16,4–8, cf. Nicomachus, *Intr. arith.* I, 7, 1–2, p. 13,9–13; cf. J.-Y. Guillaumin, "La structure du chapitre…," pp. 249–258.
[56] Probably Boethius's influence made the omission of this definition was common in the Middle Ages – see, e.g., M. Masi, "The Influence of Boethius *De Arithmetica* on Late Medieval Mathematics." In: M. Masi (Ed.), *Boethius and the Liberal Arts*, p. 82.
[57] Cf. G. Radke, *Die Theorie der Zahl…*, pp. 767–768.
[58] Boethius, *Arith.* I, 20, p. 54,74.
[59] Cf. for instance Abbo of Fleury, *In Vict. Calc*. III,4–5, p. 75–76.
[60] Boethius, *Arith.* I, 3–20, pp. 16,4–54,76, Nicomachus, *Intr. arith.* I, 7, 2–16, 10, pp. 13,13–44,7, cf. the brief interpretation by J.-Y. Guillaumin, "Boethius's *De institutione arithmetica…*," pp. 142–145. These issues are mentioned also by the most influential and best-known work of ancient mathematics, i.e., Euclid's *Elements*. Although Euclid defines (some) properties of numbers according to the listed classificatory theories as well, in many cases he adopts a different stance and follows different goals – cf. Euclid, *Elem.* VII, def. 6–14, 22, pp. 184,11–186,13, and p. 188,10–11; therefore for the medieval (at least till the 12th century) reception of arithmetic as a science, his work (although partially translated to Latin by Boethius) was not relevant and its influence on medieval arithmetic can be considered marginal in comparison to Nicomachus and in contrast to geometry where Euclid's thinking can be traced more clearly – cf. W. H. Stahl, *Roman Science. Origins, Development, and Influence to the Later Middle Ages*. Madison: University of Wisconsin Press, 1962, pp. 198–201, or M. Masi, "Arithmetic." In: D. L. Wagner (Ed.), *The Seven Liberal Arts in the Middle Ages*. Bloomington: Indiana University Press, 1983, pp. 162–164. For early medieval reception in commentary and glosses see, e.g., *De arith. Boeth.*, ad I,IIII–I,XX, pp. 136,273–139,358 or *In Boeth. Arith.* 15–40, pp. 251–255.
[61] Cf. e.g. M. Wedell, "Numbers." In: A. Classen (Ed.), *Handbook of Medieval Culture. Fundamental Aspects and Conditions of the European Middle Ages*. Vol. 2. Berlin – Boston: De Gruyter, 2015, p. 1248.
[62] Boethius, *Arith.* I, 3, p. 15,2, Nicomachus, *Intr. Arith.* I, 7, 1, p. 13,13.
[63] Martianus, *De nupt.* VII, 743, p. 269,15.
[64] Cassiodorus, *Inst.* II, 4, 2, p. 133,12.
[65] Isidorus, *Etym.* III, 3, 1, l. 1–2.
[66] The number two is problematic in many aspects as well – cf., for example, G. Radke, *Die Theorie der Zahl…*, pp. 438–446.
[67] Cf. T. Heath, *A History of Greek Mathematics.* Vol. 1: *From Thales to Euclid*. Oxford: Clarendon Press, 1921, pp. 76–84.
[68] Boethius, *Arith.* II, 4, pp. 105,3–106,21, Nicomachus, *Intr. arith.* II, 6, 1, pp. 82,10–83,11.
[69] Boethius, *Arith.* II, 4, pp. 106,22–107,43, Nicomachus, *Intr. arith.* II, 6, 2–3, pp. 83,12–85,3.
[70] Boethius, *Arith.* II, 5–39, pp. 110,22–172,41, Nicomachus, *Intr. arith.* II, 6, 4–20, 5, pp. 85,3–119,18; for a brief interpretation, see J.-Y. Guillaumin, "Boethius's *De institutione arithmetica…*," pp. 149–151); for the reception within the medieval education, see, for example, J. Høyrup, "Mathematics Education in the European Middle Ages." In: A. Karp & G. Schubring (Eds.), *Handbook on the History of Mathematics Educations*. New York: Springer, 2014, pp. 111–112; cf. also *De arith. Boeth.*, ad II,IIII–II,XXXII, pp. 144,511–147,602 or *In Boeth. Arith.* 64–81, pp. 260–263.
[71] Euclid, *Elem.* VII, def. 2, p. 184,4–5.
[72] Boethius, *Arith.* I, 3, pp. 15,2–16,1.
[73] Martianus, *De nupt.* VII, 728–729, p. 261,2–18; cf. M. Masi, "The Liberal Arts…," pp. 38–39, or W. H. Stahl, *The Quadrivium of Martianus Capella…*, p. 150.
[74] Martianus, *De nupt.* VII, 743, p. 269,15–16.
[75] Boethius, *Arith.* I, 20, p. 54,74–76, Nicomachus, *Intr. arith.* I, 17, 1, p. 44,8–9; cf. D. Albertson, *Mathematical Theologies…*, pp. 83–84.
[76] Cf. e.g. M. T. Rimple, "The Enduring Legacy of Boethian Harmony." In: N. H. Kaylor & P. E. Phillips (Eds.), *A Companion to Boethius…*, pp. 448–449, 453, or M. Masi, "The Influence of Boethius…," pp. 81–95.
[77] For more details, including the reception in the Middle Ages, see, e.g., D. Albertson, "*Boethius Noster:* Thierry of Chartres's *Arithmetica* Commentary as a Missing Source of Nicholas of Cusa's *De docta ignorantia*." *Recherches de Théologie et Philosophie médiévales* 83/1 (2016), pp. 143–199.
[78] Boethius thought of three-member number sequences. If the first sequence is labelled a, b, c and the sequence created from it a_1, b_1, c_1, then the three rules for creating inequalities can be formulated as follows:
(1) $a_1 = a$;
(2) $b_1 = a + b$;
(3) $c_1 = a + 2b + c$.
See Boethius, *Arith.* I, 32, p. 81,36–38: *"Praecepta autem tria haec sunt, ut primum numerum primo facias parem, secundum uero primo et secundo, tertium primo, secundis duobus et tertio."* English translation by M. Masi, Boethian Number Theory, p. 114: "Now these are the three rules: that you make the first number equal to the first, then put down a number equal to the first and the second, finally one equal to the first, twice the second, and the third." Cf. Nicomachus, *Intr. arith.* I, 23, 8, p. 66,15–22.
[79] Boethius, *Arith.* I, 21–II, 3, pp. 54,2–105,52, Nicomachus, *Intr. arith.* I, 17, 2–II, 5, 5. pp. 44,10–82,9; for a brief interpretation, see J.-Y. Guillaumin, "Boethius's *De institutione arithmetica…*," pp. 145–148, cf. *In Boeth. Arith.* 41–63, pp. 256–260 or *De arith. Boeth.*, ad I,XX–II,II, pp. 139,353–144,510 and for an analysis of its influence on medieval music theory, see for instance J. N. Crossley, "The Writings of Boethius and the Cogitations of Jacobus de Ispania on Musical Proportions." *Early Music History* 36 (2017), pp. 14–24.
[80] The second set of rules could be expressed as follows:
(1) $a_1 = a$;
(2) $b_1 = b - a$;
(3) $c_1 = c - (2b_1 + a)$.
See Boethius, *Arith.* II, 1, p. 94,17–31, cf. Nicomachus, *Intr. arithmeticae* II, 2, 1, pp. 74,16–75,7.
[81] Boethius, *Arith.* II, 1, pp. 93,3–96,72, Nicomachus, *Intr. arith.* II, 1, 1–2, 2, pp. 73,5–75,14.
[82] Boethius, *Arith.* II, 40–53, pp. 172,7–220,13, Nicomachus, *Intr. arith.* II, 21, 2–28, 11, pp. 120,2–144,19; for a brief interpretation, see J.-Y. Guillaumin, "Boethius's *De institutione arithmetica…*," pp. 151–153, and for reading and influence in the Middle Ages, see, for instance, *In Boeth. Arith.* 82–112, pp. 264–269; *De arith. Boeth.*, ad II,XL–II,LII, pp. 147,603–150,672 or J. N. Crossley, "The Writings of Boethius…," pp. 13–15.
[83] Boethius, Arith. II, 54, pp. 221,3–226,100, Nicomachus, Intr. arith. II, 29, 1–5, pp. 144,20–147,2; cf., for example, P. Jeserich, Musica Naturalis. Speculative Music Theory and Poetics, from Saint Augustine to the Late Middle Ages in France. Baltimore: Johns Hopkins University Press, 2013, pp. 126–134).
[84] For the first thematic transition, see Boethius, *Arith.* I, 20, p. 54,74–76, for the second transition, see ibid. II, 4, pp. 105,3–106,9, for the last transition, see ibid. II, 40, p. 172,2–7.